JANUARY, 1892.

ANNALS OF MATHEMATICS.

OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 6, Number 4.

ALL COMMUNICATIONS should be addressed to ORMOND STONE, University Station, Charlottesville, Va., U. S. A.
Entered at the Post Office as second-class mail matter.

ANNALS OF MATHEMATICS.

VOL. VI. JANUARY, 1892. No. 4.

NOTE ON ISOGONAL TRANSFORMATION; PARTICULARLY ON OBTAINING CERTAIN SYSTEMS OF CURVES WHICH OCCUR IN THE STATICS OF POLYNOMIALS.*

By DR. ROLLIN A. HARRIS, Washington, D. C.

1. If $s, = \xi + i\eta$, be a function χ of $u, = X + iY$, then if

$$f(\xi, \eta) = 0 \tag{1}$$

be the equation of any path of s,

$$f(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0$$

is the equation of its image ($\mathfrak{r}\chi$ denoting the real part of χ and $i\mathfrak{p}\chi$ the imaginary part).

2. If u be a function of $z, = x + iy$,

$$f(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0$$

may be interpreted upon either the u-plane or the z-plane; that is, it may be regarded as an equation in X, Y or in x, y.

3. We may substitute for $\chi(u)$ its derivative with respect to z (or x), and the result will still be a function of z, and so, of course, of u.

The accent will be used to denote a differentiation with respect to x. When the quantity to which it is applied is a function of z, the accent may also be regarded as denoting a differentiation with respect to that letter.

4. If two curves

$$f_1(\xi, \eta) = 0, \quad f_2(\xi, \eta) = 0 \qquad (2)\ (3)$$

intersect at certain angles in the s-plane,

$$f_1(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0,$$
$$f_2(\mathfrak{r}\chi(u), \mathfrak{p}\chi(u)) = 0$$

* An elegant exposition of the subject here referred to may be found in an article by Félix Lucas entitled Statique des polynômes: Bulletin de la Société Mathématique de France, 1889.

See also the same author's memoirs in Comptes Rendus, especially 1868 (with a review upon them in 1870) and 1888.

intersect at like angles in the u-plane (or z-plane), provided ds/du (or ds/dz) neither vanish nor become infinite at the points of intersection of (2) and (3).

5. $s = u$.

The images of the orthogonal systems

$$\xi^2 + \eta^2 = (\text{constant})^2, \tag{4}$$

$$\eta/\xi = \text{constant} \tag{5}$$

are the orthogonal systems

$$X^2 + Y^2 = (\text{constant})^2, \tag{6}$$

$$Y/X = \text{constant}. \tag{7}$$

In the z-plane,* supposing u to be a polynomial in z of order m, equation (6) denotes a system of curves of order $2m$ which have been called *lemniscates of the mth order*, or *Cassinoids*, since the continued product of the moduli of the factors of u is constant for any given curve of the system. Equation (7) denotes a system of curves of order m, each having m hyperbolic branches setting out from the zeros of u, and have been called *hyperbolas of the mth order*. All asymptotes pass through the centre of mean position of the zeros, thus dividing the z-plane into $2m$ equal angular compartments; for this reason the curves have been also called *stelloids*.

6. Regarding the zeros of the polynomial u as fixed points of unit mass which act upon the variable point inversely as their distances from it, equation (6) represents a system of *equipotential lines* in the z-plane. Upon the same suppositions, equation (7) represents a system of *lines of force*.

7. $s = \dfrac{\partial^n u}{\partial x^n}$.

The image of any curve

$$f(\xi, \eta) = 0$$

is

$$f\left[\frac{\partial^n X}{\partial x^n}, \frac{\partial^n Y}{\partial x^n}\right] = 0. \tag{8}$$

In particular, this shows that because the systems (4) and (5) are orthogonal, so are the systems

$$\left[\frac{\partial^n X}{\partial x^n}\right]^2 + \left[\frac{\partial^n Y}{\partial x^n}\right]^2 = (\text{constant})^2, \tag{9}$$

$$\frac{\partial^n Y}{\partial x^n} \Big/ \frac{\partial^n X}{\partial x^n} = \text{constant}. \tag{10}$$

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* Journal de l'École Polytechnique, 1879: Géométrie des polynômes.
Holzmüller, Theorie der isogonalen Verwandschaften, S. 202, 204 *), 205 **).
Annals of Mathematics, Vol. iv, p. 73.

If $n = 1$, the curves (9) become

$$X'^2 + Y'^2 = (\text{constant})^2, \tag{11}$$

which are *lines of equal expansion* for the function u; the orthogonal trajectories are

$$Y'/X' = \text{constant}. \tag{12}$$

8. $s = \dfrac{\partial^n \log u}{\partial x^n}$.

The image of any curve

$$f(\xi, \eta) = 0$$

is

$$f\left[\frac{\partial^n \log R}{\partial x^n}, \frac{\partial^n \theta}{\partial x^n}\right] = 0. \tag{13}$$

When $n = 1$, s becomes u'/u, and (13)

$$f(\log' R, \theta') = 0.$$

Equation (4) now transforms into

$$\frac{X'^2 + Y'^2}{X^2 + Y^2} = (\text{constant})^2. \tag{14}$$

These curves are *isodynamic lines*, upon the hypotheses made in § 6. $\xi, -\eta$ are equal to the total component forces along the x- and the y-axis respectively.

From equation (5) it follows that the system

$$\frac{XY' - X'Y}{XX' + YY'} = \text{constant} \tag{15}$$

is orthogonal to (14). These curves are *lines of parallel action.*

Lucas's generalization of Rolle's theorem. Since

$$\xi + i\eta = \frac{u'}{u}$$

it is clear that a variable point in the z-plane will be in equilibrium when, and only when, it coincides with a zero of the derived equation $u' = 0$.

Every closed convex contour surrounding the roots of an algebraic equation surrounds also the roots of the derived equation.

9. *Application to Hydrodynamics.* A steady irrotational motion is supposed to take place in planes parallel to xy.

Let $X =$ constant be the lines of equal velocity-potential; then $Y =$ constant are the stream-lines, (11) the lines of equal velocity, (12) the lines along which the direction of flow is constant.

Here and in § 7, the only condition imposed upon u is that it be a function of z; it may, therefore, be replaced by any given function of u, and the four systems of curves just mentioned will be altered accordingly. If $\log u$ be substituted for u, these systems assume the forms (6), (7), (14), and (15), respectively. Suppose, in a fluid of unlimited extent, line sources, each of the same strength and perpendicular to the plane xy, to pass through the zeros of a polynomial u. For this motion (6) are the lines of equal velocity-potential, (7) the stream-lines, (14) the lines of equal velocity, and (15) the lines along which the direction of flow is constant.

ON THE PERMANENCE OF EQUIVALENCE.

[From the Discussions of the Mathematical Club, Cornell University.]

The so-called principle of the Permanence of Equivalent Forms was stated by a member as follows: If two symbolic expressions be equivalent when the symbols are general in form but restricted to a certain class of values, they will continue to be equivalent when the symbols are wholly unrestricted in value as well as in form.

It was pointed out that the principle as thus stated might better be called the Permanence of the Equivalence of Forms; and that the author* who gives it the greatest prominence uses the principle chiefly as a basis for interpreting new symbols,—as when the forms $a^m \times a^n$ and a^{m+n} are made permanently equivalent by convention, and thence is derived the interpretation of the symbol a^m when m is fractional or negative; and that he professes to apply the principle only to cases in which the fundamental definitions of the symbols cease to be applicable, thus disclaiming its validity in removing the restrictions from such a formula as

$$\sin(\theta + \theta') = \sin\theta \cos\theta' + \cos\theta \sin\theta',$$

supposed to be proven for positive acute angles.

It was noticed, however, that Peacock himself in proving the general Binomial Theorem by Euler's method uses the principle in question to demonstrate the permanence of the fundamental equation

$$f(m) \times f(n) = f(m + n),$$

wherein $f(m)$ stands for the "binomial expansion"

$$1 + mx + \frac{m(m-1)}{1\,.\,2}x^2 + \ldots,$$

thus pushing the principle beyond the field of convention into that of logic, since here no question of interpreting new symbols is involved; and that this illustrates the need of some demonstrated and more sharply-defined principle applicable to such cases.

This need is likewise suggested by the fact that Euler disclaims the universality of the principle of the Permanence of Equivalence, citing as an instance of its failure the equation

$$m = \frac{1-a^m}{1-a} + \frac{(1-a^m)(1-a^{m-1})}{1-a^2} + \frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3} + \ldots,$$

* Peacock's Symbolic Algebra.

which is true only when m is an integer; so that Peacock charges him with denying the generality of the only principle that could give validity to his proof of the Binomial Theorem.

It was also remarked that although the principle of permanence is most plausible when applied to ordinary algebraic polynomials, yet this plausibility is weakened in the present case by the fact that the expressions $f(m)$, $f(n)$, $f(m+n)$, taken as wholes, change their character in one important particular when m, n take their unrestricted values: viz., that $f(m)$, which consists of a finite number of terms when m is a positive integer, becomes an infinite series in all other cases, and that moreover in certain cases $f(m)$ and $f(n)$ might both be infinite series, and yet $f(m+n)$ be a finite series.

In view of the wide-spread impression that Euler's proof of the Binomial Theorem is wanting in logical rigor, some of the members had sought for an algebraic principle that should fill out what was probably in Euler's mind. The purpose was served by the "Principle of Algebraic Identity," which is a special case of the permanence of equivalence, and which it is convenient here to enunciate separately for the cases of one and of two variables:—

(a) If there be two rational integral functions involving m in degrees not exceeding r, and if these functions be equal for more than r separate values of m,—then they are equal for all values of m; i. e., they are identical and the coefficients of corresponding terms are equal.

(b) If there be two rational integral functions, each involving m in degrees not exceeding r, and n in degrees not exceeding s, and if there be certain definite values of n, more than s in number, such that when n has any one of these values the functions are equal for more than r separate values of m,—then the functions are equal for all values of m and n, and the coefficients of corresponding terms are equal.

The first of these principles is derived, in most works on algebra, from the theorem that a function of the rth degree cannot vanish for more than r separate values of its variable. It can be extended without difficulty to functions of two variables as enunciated in the second principle, and also to functions of more than two variables if desired.

These principles were applied to establish the permanence of the equivalence in question as follows: Suppose the series $f(m)$ to be actually multiplied by $f(n)$, and the product, arranged in ascending powers of x, to be compared with the series $f(m+n)$ similarly arranged. It appears by inspection that the first two or three terms correspond; and it remains to be shown that the general terms also correspond. For let the coefficients of x^r in $f(m) \times f(n)$ and in $f(m+n)$, respectively, be denoted by $\varphi_r(m, n)$, $\theta_r(m, n)$; then the functions φ_r, θ_r involve m and n in degrees not exceeding r, and are numerically

equal for every positive integral value of m and n; hence, by the second principle, $\varphi_r(m, n)$ and $\theta_r(m, n)$ are equal for all values of m and n, and are algebraically identical. But these represent any two corresponding coefficients in the two series in question, hence the series $f(m) \times f(n)$ and $f(m+n)$ are identical term by term.

Thus the permanence of Euler's fundamental equivalence was established, from which the proof of the general Binomial Theorem easily follows.*

It was pointed out that Euler's is to be regarded as the most philosophical proof of the Binomial Theorem, growing as it does by a natural and general method out of the result for a positive integer, which itself rests naturally on the theory of combinations.

Brief reference was also made to a further extension of the same theorem, in which the terms of the binomial are replaced by general functional symbols of operation, the exponents indicating repetitions of these operations; and it was shown that the extension is valid only when the symbols follow the same commutative, associative, and distributive laws that lie at the basis of multiplication; these laws are fulfilled by imaginary numbers regarded as versi-tensors,† and by many operators in the method of differences, finite and infinitesimal; e. g. the operator $\left[a\dfrac{d}{dx} + b\dfrac{d}{dy}\right]^n$, acting on any function of x and y, is equivalent to the operator

$$a^n \frac{d^n}{dx^n} + na^{n-1}b\frac{d^n}{dx^{n-1}dy} + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2\frac{d^n}{dx^{n-2}dy^2} + \ldots .$$

In such cases, where there is no interpretation of new symbols, the validity of the result does not depend on the general principle of permanence, but on some special statement of it which determines what particular properties of the restricted symbols are to be preserved in the more general ones.

On the other hand, where there are new symbols to be interpreted, it is usual to fix on some controlling equivalence or equivalences which are to remain permanent, and in accordance with which the new symbols are to be assigned their meaning. Of this process the two following examples were given:—

(1) The interpretation of x^m, when m is imaginary, is obtained by making permanent the exponential expansion

$$x^m = 1 + \lambda m + \frac{\lambda^2 m^2}{1\,.\,2} + \ldots , \qquad [\lambda = \log_e x$$

whence x^{a+bi} is to be equivalent to the operator $x^a(\cos \lambda b + i \sin \lambda b)$, whose tensor is x^a, and versorial angle b radians; and the more general symbol

* See Hall and Knight's, or Todhunter's Algebra.

† Oliver, Wait, and Jones's Algebra, p. 264 *et seq.*

$(x+yi)^{a+bi}$, to the operator whose tensor is $r^a e^{-b\theta}$, and versorial angle $a\theta + b\lambda$, where r and θ are the tensor and versorial angle of $x+yi$, and $\lambda = \log_e r$. This interpretation preserves the "laws of exponents," and includes the ordinary meanings as special cases. The Binomial Theorem may then be correspondingly extended by establishing the algebraic identity of the two series

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots,$$

$$1 + \lambda m + \frac{\lambda^2 m^2}{1.2} + \ldots,$$

where $\lambda = \log_e(1+x) = x - \frac{x^2}{2} + \ldots$, the two series being known to be equivalent for all real values of x and m that make them both convergent.

(2) An interpretation of $\left[\frac{d}{dx}\right]^m$ when m is fractional or imaginary,* may be derived by means of the equivalence

$$\left[\frac{d}{dx}\right] \cdot \left[\frac{d}{dx}\right]^n \cdot u = \left[\frac{d}{dx}\right]^{m+n} \cdot u,$$

in conjunction with one of the auxiliary equations

$$\left[\frac{d}{dx}\right]^r \cdot e^{ax} = a^r \cdot e^{ax},$$

$$\left[\frac{d}{dx}\right]^r \cdot x^p = \frac{\Gamma(p+1)}{\Gamma(p-r+1)} x^{p-r},$$

in which the "gamma function" $\Gamma(p+1)$ is identical with $p!$ when p is a positive integer.

The whole discussion showed that the general principle of permanence should be limited to the field of interpretation; and that beyond that field we need to give a precise and special statement to it, and to the nature of the restrictions whose unimportance it asserts; such special statement requiring demonstration from the properties of the symbols involved.

Another aspect of this question will be treated of in connection with the "Principle of Continuity," which will be the subject of a future discussion.

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* See De Morgan's Calculus, p. 599, with references; Kelland on "General Differentiation," in Edinburgh Trans., Vol. XIV, with interesting applications; also Centre's papers in Cambridge and Dublin Math. Journal, Vols. III, IV, V.

ON BESSEL'S FUNCTIONS OF THE SECOND KIND.

By Dr. Maxime Bôcher, Cambridge, Mass.

In the first volume of the Mathematische Annalen* (1869) Hankel has given an investigation of some of the properties of Bessel's functions. This paper is probably unknown or inaccessible to many who frequently have to make practical use of these functions in physics, so that I have thought it worth while to reproduce one of the important results contained in it† accompanying it with a few critical remarks.

Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

is found (cf. Forsyth,‡ p. 158) to have as a particular solution, the following series, which is convergent for all values of x whatever value n may have:

$$J_n(x) = \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(n+p+1)\,.\,\Gamma(p+1)} \left[\frac{x}{2}\right]^{2p}.$$

Here $\Gamma(n)$ is the Eulerian integral which reduces to $(n-1)!$ when n is a positive integer, and which is extended to negative values of n in any legitimate way, for instance by means of the formula $\Gamma(n+1) = n\,\Gamma(n)$.§

Thus defined, $J_n(x)$ is called a Bessel's function of x of the first kind of order n. Since the parameter n enters into the differential equation only through its square, $J_{-n}(x)$ will also be a solution of the equation, and in general the complete primitive of Bessel's equation will have the form $AJ_n(x) + BJ_{-n}(x)$. If however n is a real integer, the relation $J_n(x) = (-1)^n J_{-n}(x)$ holds true (Forsyth, p. 160); so that in this case the above solution of Bessel's equation is no longer general. Confining our attention then, as we may do without loss of generality, to *positive* integral values of n we have the problem: *To find in this case a solution of Bessel's equation linearly independent of* $J_n(x)$.

If the parameter of the equation were $n - \Delta n$ we should have as two independent solutions

$$J_{n-\Delta n}(x) \text{ and } J_{-n+\Delta n}(x).$$

* p. 467. *Die Cylinderfunctionen erster und zweiter Art.*

† I have confined myself to § 1, although the following sections, especially the last two, are of the greatest interest to the physicist as well as to the student of the theory of functions.

‡ "A Treatise on Differential Equations." I shall frequently refer to this book by merely mentioning the author's name.

§ For the purpose of this note we may confine our attention to *real* values of n. This restriction however is in general undesirable and is not made by Hankel.

Any entire linear combination of these functions will accordingly also be a solution. Such a combination is

$$\frac{(-1)^n J_{-n+\Delta n}(x) - J_{n-\Delta n}(x)}{2\Delta n}.$$

Adding to the numerator of this fraction the null expression $J_n(x) - (-1)^n J_{-n}(x)$ will not affect it, so that we may write our solution

$$\frac{(-1)^{n+1}\left[J_{-n}(x) - J_{-n+\Delta n}(x)\right] + J_n(x) - J_{n-\Delta n}(x)}{2\Delta n}.$$

If, now, we let Δn approach zero, we obtain

$$K_n(x) = \tfrac{1}{2}\left[\frac{dJ_n(x)}{dn} + (-1)^{n+1}\frac{dJ_{-n}(x)}{dn}\right]$$

as a second solution of Bessel's equation *when n is a positive integer.* This function is called a Bessel's function of the second kind of order n.

Now we obtain at once by differentiation

$$\frac{dJ_n(x)}{dn} = \log\left[\frac{x}{2}\right].\,J_n(x) + \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty}\frac{(-1)^p}{\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}\frac{d[\Gamma(n+p+1)]^{-1}}{dn},$$

$$\frac{dJ_{-n}(x)}{dn} = -\log\left[\frac{x}{2}\right].\,J_{-n}(x) + \left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=\infty}\frac{(-1)^p}{\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}\frac{d[\Gamma(-n+p+1)]^{-1}}{dn}.$$

In order to differentiate the reciprocals of $\Gamma(n+p+1)$ and $\Gamma(-n+p+1)$ with respect to n we will introduce a new function using a notation similar to that of Gauss, and write

$$\psi(x) = \frac{d\log\Gamma(x)}{dx}.$$

This function ψ will evidently have the property

$$\psi(x) + \frac{1}{x} = \psi(x+1);$$

so that if x is a positive integer,

$$\psi(x) = \frac{1}{x-1} + \frac{1}{x-2} + \dots + \tfrac{1}{3} + \tfrac{1}{2} + 1 + \psi(1),$$

and $\psi(1)$ has approximately the value -0.5772. In terms of this function we shall have

$$\frac{d\Gamma(x)}{dx} = \Gamma(x).\,\psi(x),$$

and hus we get

$$\frac{dJ_n(x)}{dn} = \log\left[\frac{x}{2}\right] . J_n(x) - \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty} \frac{(-1)^p . \psi(n+p+1)}{\Gamma(n+p+1) . \Gamma(p+1)} \left[\frac{x}{2}\right]^{2p}.$$

The expression for the derivative of $J_{-n}(x)$ we might of course get in the same way, but this would introduce the factor $\frac{\psi(-n+p+1)}{\Gamma(-n+p+1)}$ which has for positive integral values of n and p the indeterminate form $\frac{\infty}{\infty}$ as long as $p \overline{\gtrless} n-1$. We will therefore divide our series into two parts at this point, getting

$$\frac{dJ_{-n}(x)}{dn} = -\log\left[\frac{x}{2}\right] . J_{-n}(x) + \left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{(-1)^p}{\Gamma(p+1)} \left[\frac{x}{2}\right] \frac{d[\Gamma(-n+p+1)]^{-1}}{dn}$$

$$+ \left[\frac{x}{2}\right]^{-n} \sum_{p=n}^{p=\infty} \frac{(-1)^p \psi(-n+p+1)}{\Gamma(p+1) . \Gamma(-n+p+1)} \left[\frac{x}{2}\right]^{2p}.$$

In order to perform the differentiation in the cases which still remain resort must be had to a simple formula in the theory of the Γ function; namely,

$$\Gamma(x) . \Gamma(1-x) . \sin x\pi = \pi ;$$

from which we get

$$\frac{1}{\Gamma(-n+p+1)} = \frac{1}{\pi} \sin(-n+p+1)\pi . \Gamma(n-p),$$

and therefore,

$$\frac{d}{dn}\left[\frac{1}{\Gamma(-n+p+1)}\right] = -\cos(-n+p+1)\pi . \Gamma(n-p)$$

$$-\frac{1}{\pi}\sin(-n+p+1)\pi . \Gamma(n-p) . \psi(n-p).$$

The value of this expression when n and p are positive integers and $p < n$ is

$$-(-1)^{-n+p+1} . \Gamma(n-p) = -(-1)^{n-p+1} . \Gamma(n-p).$$

Thus we have finally performed the differentiation indicated above. The series we get may however be written in a more convenient form if in the second (and infinite) part the new variable of summation $q = p - n$ be introduced. This gives us

$$\frac{dJ_{-n}(x)}{dn} = -\log\left[\frac{x}{2}\right] . J_{-n}(x) - (-1)^{n+1} \left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{\Gamma(n-p)}{\Gamma(p+1)} \left[\frac{x}{2}\right]^{2p}$$

$$-(-1)^{n+1} \left[\frac{x}{2}\right]^n \sum_{q=0}^{q=\infty} \frac{(-1)^q \psi(q+1)}{\Gamma(n+q+1) . \Gamma(q+1)} \left[\frac{x}{2}\right]^{2q}.$$

We thus get finally as the development of our "Bessel's function of the second kind of order n" (where n of course represents a positive integer):

$$K_n(x) = \log\left[\frac{x}{2}\right] . J_n(x) - \tfrac{1}{2}\left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{\Gamma(n-p)}{\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}$$

$$-\tfrac{1}{2}\left[\frac{x}{2}\right]^{n} \sum_{p=0}^{p=\infty} \frac{(-1)^p\left[\psi(n+p+1)+\psi(p+1)\right]}{\Gamma(n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}.$$

I have now completed the account (somewhat fuller, and I hope on that account more easy to read, than the original) which I wished to give of § 1 of the paper of Hankel quoted above.*

In connection with this subject the following points may be mentioned:—

(1) That the series we have obtained for $\frac{dJ_n(x)}{dn}$ and $\frac{dJ_{-n}(x)}{dn}$ converge unconditionally may be seen at once by forming the quotient of the $(p+1)$ st to the pth term, as this quotient will be seen at once to become and remain as small as we please for any finite value of x if we take the number of terms p large enough. Nevertheless, we took one step above which still requires justification. We assumed, namely, that the derivatives with respect to n of the series

$$-\sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p} \text{ and } \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(-n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2}$$

could be found by differentiating term by term. This, however, requires proof as there are many series (for instance, many trigonometric series) which cannot be so differentiated.

Now it is well known that a series may be differentiated term by term if the series obtained by performing the differentiation in this way converges *uniformly* for such values of the variable as we areconsidering. We obtained from the first of the above-written series by differentiating term by term the series:

$$-\sum_{p=0}^{p=\infty} \frac{(-1)^p\psi(n+p+1)}{\Gamma(n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p};$$

and the second of the above series gave us, if we neglect a finite number of terms which need not concern us here,

$$-(-1)^{n+1}\sum_{p=0}^{p=\infty} \frac{(-1)^p\psi(p+1)}{\Gamma(n+p+1)\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}.$$

* The closing part of this section (quoted by Forsyth, p. 165, Ex. 1) of which the above may be considered a special case, would seem to be of less general interest.

These are the series which must be proved uniformly convergent for all positive values of n. This we can do at once by means of the theorem :*

When the series of the numerically greatest values assumed by the terms of the series in the interval in question converges, the series is uniformly convergent.

The terms will, however, all become numerically greatest for $n = 0$, and for this, as well as for every other value of n, the series have been seen to converge.

(2) The function $J_n(x)$ satisfies for all values of n the fundamental relations, which might be called with Gauss *relationes inter contiguas:*

$$nJ_n(x) = \tfrac{1}{2}\, x\, [J_{n+1}(x) + J_{n-1}(x)\,],$$

$$\frac{dJ_n(x)}{dx} = \tfrac{1}{2}\, [J_{n-1}(x) - J_{n+1}(x)\,].$$

In fact this seems to be the reason† why the coefficient of x^n in the development of $J_n(x)$ is taken as $[2^n \Gamma(n+1)\,]^{-1}$; for if it were given a simpler value, these relations would not be satisfied.

It is accordingly of the highest practical importance, that the new function $K_n(x)$ should also satisfy similar relations; and from our method of introducing this function it is evident that it will do so. For we have

$$(n - \Delta n)\, J_{n-\Delta n}(x) = \tfrac{1}{2}\, x\, [J_{n+1-\Delta n}(x) + J_{n-1-\Delta n}(x)\,],$$

$$-(n - \Delta n)\, J_{-n+\Delta n}(x) = \tfrac{1}{2}\, x\, [J_{-n+1+\Delta n}(x) + J_{-n-1+\Delta n}(x)\,].$$

From these equations we get

$$(n - \Delta n)\, \frac{J_{n-\Delta n}(x) - (-1)^n J_{-n+\Delta n}(x)}{2\Delta n} = \tfrac{1}{2}\, x \left[\frac{J_{n+1-\Delta n}(x) - (-1)^{n+1} J_{-n-1+\Delta n}(x)}{2\Delta n} + \frac{J_{n-1-\Delta n}(x) - (-1)^{n-1} J_{-n+1+\Delta n}(x)}{2\Delta n}\right].$$

Or, in the limit, when $\Delta n = 0$,

$$nK_n(x) = \tfrac{1}{2}\, x\, [K_{n+1}(x) + K_{n-1}(x)\,].$$

The other fundamental relation can be proved in exactly the same way.

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* cf. Harnack: Die Elemente der Differential- und Integralrechnung. Leipzig, 1881. Translated by G. L. Cathcart, 1891. Book III, Chap. IV, should be consulted.

† I mean the reason which is of weight with practical physicists. Mathematically there are several other reasons for this choice, which, however, have nothing to do with the definition of $J_n(x)$ as a solution of a differential equation.

(3) We may, however, find another second solution of Bessel's equation satisfying these relations by adding to $K_n(x)$ defined as above $AJ_n(x)$, where A is a function neither of n nor of x.* For practical purposes it will be advisable to simplify $K_n(x)$ as much as possible by this means. The first term of $K_n(x)$ is $\log(x/2) \, . \, J_n(x)$ which may be reduced in this way to $\log x \, . \, J_n(x)$. We have already noticed that when x is a positive integer,

$$\psi(x) = \frac{1}{x-1} + \frac{1}{x-2} + \ldots + \tfrac{1}{3} + \tfrac{1}{2} + 1 + \psi(1).$$

The constant $\psi(1)$ may, however, be omitted from all our ψ's; since it is readily seen that by doing so we merely add $\psi(1) \, . \, J_n(x)$ to $K_n(x)$. We can then write our new solution, which I will denote by $\{K_n(x)\}$, in the following form, hardly less adapted to numerical calculation than the ordinary development of $J_n(x)$:

$$\{K_n(x)\} = \log x \, J_n(x) - \tfrac{1}{2}\left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{(n-p-1)!}{p!}\left[\frac{x}{2}\right]^{2p}$$

$$-\tfrac{1}{2}\left[\frac{x}{2}\right]^{n} \sum_{p=0}^{p=\infty} \frac{(-1)^p}{(n+p)!\,p!}\left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{p} + 1 + \tfrac{1}{2} + \ldots + \frac{1}{n+p}\right]\left[\frac{x}{2}\right]^{2p}.$$

(4) Finally it remains to mention the other forms in which the solution $K_n(x)$ is usually found,† namely, a series proceeding either wholly or in part according to Bessel's functions of the first kind $J_p(x)$, the positive integer p being the variable of summation. On the contrary Hankel's series is merely a power series‡ except for the necessary logarithmic terms. Hankel's expression for $K_n(x)$ is therefore the most elementary in form, and I think no one who is familiar with the methods of obtaining the other expressions for $K_n(x)$ will deny that Hankel's expression is also far more easily found. Moreover, his has the advantage, as all students of the modern theory of linear differential equations know, of being a general method, applicable to the large class of cases where logarithmic terms occur in the solution. Why should the working physicist not have the advantage of this simple and in the highest degree practical theory?

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* Of course we might also multiply $K_n(x)$ by any factor not involving n or x.

† *C. Neumann:* Theorie der Besselschen Functionen. 1867.

Lommel: Studien über die Besselschen Functionen. 1868. Also *Forsyth*, § 105.

‡ The function $J_n(x)$ which occurs in the above expression for $K_n(x)$ could of course be expressed as a power series too, if desired. The above form, however, will probably be as convenient in most cases, as $K_n(x)$ is rarely used except when $J_n(x)$ has also to be computed.

POCKELS ON THE DIFFERENTIAL EQUATION $\Delta u + k^2 u = 0$.*

Although this work might appear from its title, even to the mathematician or physicist, to have such a purely technical character as to make it of interest to a few specialists only, yet to one acquainted with the present condition of the mathematical sciences in Germany even the title-page will offer the promise of interest. A few introductory lines only are from the pen of Professor Klein, but the whole work is imbued with his ideas and methods. Professor Klein has distinguished himself in the past by work in many branches of pure mathematics; but up to the present time, with the exception of one small pamphlet, he has left the science of mathematical physics almost untouched in his published works. Breadth of view and largeness of conception are qualities which are too often sacrificed by the specialist in order that he may give greater thoroughness in details. For this reason the appearance of an outsider, as it were, in the field of mathematical physics should be looked upon by all interested in the real progress of this science as a rare piece of good fortune. For there is serious danger now threatening many branches of scientific study, that through too great specialization by small bodies of investigators they may become isolated from the rest of the scientific movement of the age. Just as the human race will degenerate wherever marriage takes place for several generations between the members of a few families only, so every little band of scientific specialists needs a constant intercourse with other investigators not so closely related to them in thought, if their work is not to become arid and comparatively worthless.

But if Professor Klein has turned his attention to mathematical physics during the last few years, it has not been for the purpose of enriching this branch of science with the methods and results of pure mathematics, although this will undoubtedly be the effect if physicists will come to meet him half way. On the contrary he has been moved by the desire of increasing his own power and that of all pure mathematicians who will stand by him, by making his own the methods of mathematical physics, so different in many respects from those of pure mathematics. The ideas he has here found come, in great part, from the English physicists: Maxwell, Thomson, Rayleigh, and would have repelled many pure mathematicians by their laxity of proof. They have in them however a strength which comes from their keeping in touch with the world of the senses, and which makes them of unequaled value to one seeking to discover new truths. In physics, equations appear at every step which no mathemati-

* Ueber die partielle Differentialgleichung $\Delta u + k^2 u = 0$ und deren Auftreten in der mathematischen Physik, von Friedrich Pockels, mit einem Vorwort von F. Klein. Teubner. Leipzig, 1891.

cian can solve ; but the observer of nature can often tell from what he sees going on before him some properties which the solutions of these equations must necessarily have. It is this principle which has been borrowed by Professor Klein from the English physicists, and which gives the work now under consideration an interest even to the student of pure mathematics.

The equation $\Delta u + k^2 u = 0$ is of almost paramount importance in physics, second in importance perhaps only to Laplace's equation $\Delta u = 0$, and Dr. Pockels considers its applications to all the different branches of physics in which it comes up, chief among these being the question of small vibrations of elastic bodies. The quickness with which he passes from one branch of physics to another, dwelling on none very long but returning frequently to each, is very characteristic of the book. In fact we have here, less an application of mathematical analysis to a physical problem, than the study of a single analytical subject from as many different physical points of view as possible; rather physical mathematics than mathematical physics.

The partial differential equation with which we are here concerned has, in the last decades, been the subject of numerous memoirs which are scattered through the mathematical journals of various countries. The main purpose of Dr. Pockels's book is to collect what is now known on the subject ; but it is no mere compilation even aside from its not unimportant original contributions to the theory ; for in it the old results are woven into a beautiful new fabric which will be a help to the reader and above all will enlarge his ideas. And it is not unimportant to insist upon this in these days when we have seen books written, as this one was, for the purpose of collecting the results in a modern branch of mathematics, which have been little more than the more or less direct reprint of papers by first one author and then another. There is a real need for books in the more modern branches of mathematics, but they must not be merely a patchwork of what can already be found as well elsewhere.

The application of physics to mathematics is the thought which runs through Dr. Pockels's whole book. It is even the cause of what seems to us to be an imperfection. The three dimensions of the space to which we seem unfortunately to be confined, have no analogy in pure analysis. It would have been possible to have indicated, at least to some extent, the extension of the results found to the case where the differential parameter Δ contains any number of variables instead of, at most, three. We may be sure that this would have been done if the work had proceeded directly from Professor Klein's pen. But Dr. Pockels is essentially a physicist and we ought perhaps to be satisfied with what he has given us, as he has often dwelt upon points which most physicists would have discarded as too purely mathematical.

MAXIME BÔCHER.

AN APPLICATION OF ELLIPTIC FUNCTIONS TO A PROBLEM IN GEOMETRY.

By Prof. Jas. H. Boyd, St. Paul, Minn.

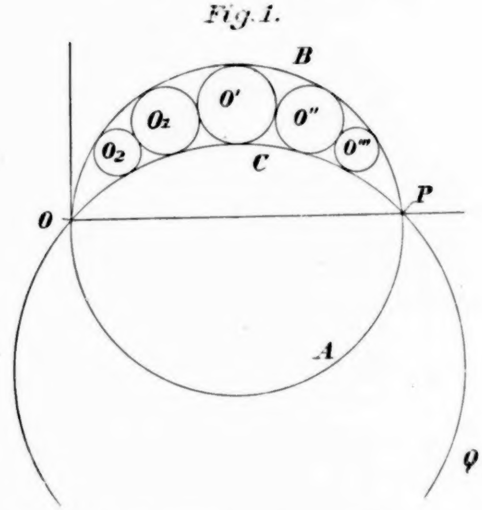
In one of the crescents formed by two intersecting circles, smaller circles are inscribed touching each other successively. Required (1) the sum of the radii of the inscribed circles; (2) the sum of their areas.

I.

The circles OBP and OCQ (Fig. 1), or

$$\left.\begin{array}{l} x^2 + y^2 + bx + cy = 0 \\ x^2 + y^2 + b'x + c'y = 0 \end{array}\right\}, \tag{1}$$

Fig. 1.



after reciprocation by means of the transformation

$$x = \frac{k^2 X}{X^2 + Y^2} \text{ and } y = \frac{k^2 Y}{X^2 + Y^2}, \tag{2}$$

O being the centre of reciprocation, transform into the straight lines (Fig. 2), OST and $OS'T'$, or

$$\left.\begin{array}{l} b\,X + c\,Y + k^2 = 0 \\ b'X + c'Y + k^2 = 0 \end{array}\right\},$$

which make the same angle with each other as the two circles OAP and OCQ.

The circles . . . , O_2, O_1, O', O'', O''', . . . of Fig. 1 are transformed into the circles . . . , O_1, O', O'', . . . of Fig. 2, touching the straight lines OST and $OS'T'$. Conversely any two straight lines OST and $OS'T'$ inclosing the same angle that any two given intersecting circles do may be transformed into the latter by a proper choice of the centre of reciprocation.

We therefore attack the given problem as follows: Draw the lines OST, $OS'T'$ (Fig. 2) making with each other an angle equal to that formed by the given circles (Fig. 1).

Let z^2 be the radius of any arbitrary circle inscribed in the angle $TSOS'T'$ and let OO' be the bisector of this angle. Call the angle TOO', a. Then

$$TO'' = z^2 \frac{1 + \sin a}{1 - \sin a} = z^2 h^{-2},$$

where h^2 is put for $\dfrac{1 - \sin a}{1 + \sin a}$ (which is < 1).

The radii of the series of inscribed circles . . . , O_1, O', O'', . . . will therefore be

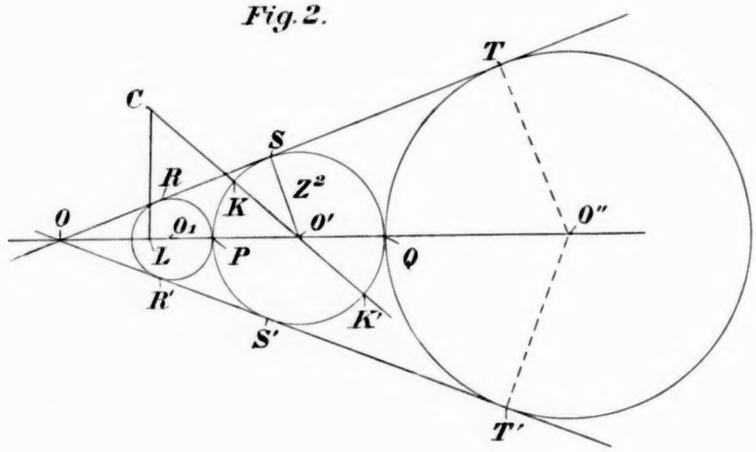
$$\ldots,\ h^4 z^2,\ h^2 z^2,\ z^2,\ h^{-2} z^2,\ h^{-4} z^2,\ \ldots. \tag{3}$$

In Fig. 2, from any point C draw CL perpendicular to OO', meeting OO' in some point L. Put $OL = a$ and $CL = b$. Draw a straight line through C and O' meeting the circumference of O' in K and K'.

Since

$$CK = \sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} - z^2,$$

Fig. 2.



and

$$CK' = \sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} + z^2,$$

therefore, after reciprocation with respect to the centre C, the radius of the circle in the crescent (Fig. 1) which corresponds to the circle O' of radius z^2, will be equal to

$$\tfrac{1}{2}\left[\frac{1}{\sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} - z^2} - \frac{1}{\sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} + z^2}\right].$$

or

$$\frac{\tan^2 a \,.\, z^2}{(z^2 - z_1^2)(z^2 - z_2^2)}, \tag{4}$$

where z_1^2 and z_2^2 are the roots of the equation

$$(z^2)^2 - \frac{2a}{\cos a}(z^2) + (a^2 + b^2)\tan^2 a = 0,$$

considered as a quadratic equation.

In this equation a is given, being one-half the angle which the two given circles make with each other. a and b are also known since they are determined by the condition that the centre C (Fig. 2) of reciprocation was so chosen that the two straight lines OST and OST' transform into the two circles OCQ and $OAPB$ (Fig. 1).

Hence the radius of the reciprocal of any nth circle will be

$$\frac{\tan^2 a \,.\, h^{\pm 2n} z^2}{(h^{\pm 2n} z^2 - z_1^2)(h^{\pm 2n} z^2 - z_2^2)}.$$

Therefore the sums of the radii and areas of the circles in the crescent will be respectively,

$$S_r(z) = \sum_{+\infty}^{-\infty}{}^{n} \frac{\tan^2 a \,.\, h^{2n} z^2}{(h^{2n} z^2 - z_1^2)(h^{2n} z^2 - z_2^2)}, \tag{5}$$

$$S_A(z) = \sum_{+\infty}^{-\infty}{}^{n} \frac{\pi \tan^4 a \,.\, h^{4n} z^4}{(h^{2n} z^2 - z_1^2)^2 (h^{2n} z^2 - z_2^2)^2}. \tag{6}$$

II.

In this section we shall calculate $S_r(z)$ in terms of σ-functions.

From § 8, (6) of Weierstrass and Schwarz's Elliptic Function Formulas (Edition 1885), we have

$$\frac{\sigma'}{\sigma}(u) = \frac{\eta}{\omega} u + \frac{\pi i}{2\omega}\left[\frac{z^2 + 1}{z^2 - 1} + \sum{}^{n} \frac{2h^{2n} z^{-2}}{1 - h^{2n} z^{-2}} - \sum{}^{n} \frac{2h^{2n} z^2}{1 - h^{2n} z^2}\right], \tag{7}$$

where, according to Prof. Schwarz's notation,

$$z = e^{\frac{u\pi i}{2\omega}}, \quad h = e^{\frac{\omega'\pi i}{\omega}};$$

and we put, analogously,

$$z_1 = e^{\frac{u_1\pi i}{2\omega}} \text{ and } z_2 = e^{\frac{u_2\pi i}{2\omega}};$$

whence

$$\frac{z}{z_1} = e^{\frac{(u-u_1)\pi i}{2\omega}} \text{ and } \frac{z}{z_2} = e^{\frac{(u-u_2)\pi i}{2\omega}};$$

that is, when u becomes $u - u_1$ or $u - u_2$, z becomes $\frac{z}{z_1}$ or $\frac{z}{z_2}$ respectively.

Hence,

$$\frac{\sigma'}{\sigma}(u-u_1) = \frac{\eta}{\omega}(u-u_1) + \frac{\pi i}{2\omega}\left[\frac{z^2+z_1^2}{z^2-z_1^2} + \sum{}^n \frac{2h^{2n}z^{-2}}{z_1^{-2}-h^{2n}z^{-2}} - \sum{}^n \frac{2h^{2n}z^2}{z_1^2-h^{2n}z^2}\right], \tag{8}$$

$$\frac{\sigma'}{\sigma}(u-u_2) = \frac{\eta}{\omega}(u-u_2) + \frac{\pi i}{2\omega}\left[\frac{z^2+z_2^2}{z^2-z_2^2} + \sum{}^n \frac{2h^{2n}z^{-2}}{z_2^{-2}-h^{2n}z^{-2}} - \sum{}^n \frac{2h^{2n}z^2}{z_2^2-h^{2n}z^2}\right]. \tag{9}$$

Subtracting (9) from (8)

$$\frac{\sigma'}{\sigma}(u-u_1) - \frac{\sigma'}{\sigma}(u-u_2) = -\frac{\eta}{\omega}(u_1-u_2) + \frac{\pi i(z_1^2-z_2^2)}{\omega}\sum_{+\infty}^{-\infty}{}^n \frac{h^{2n}z^2}{(h^{2n}z^2-z_1^2)(h^{2n}z^2-z_2^2)}.$$

After transposing $-\frac{\eta}{\omega}(u_1-u_2)$ and dividing by $\frac{\pi i(z_1^2-z_2^2)}{\omega\tan^2\alpha}$ we obtain

$$S_r(z) = \sum_{+\infty}^{-\infty}{}^n \frac{\tan^2\alpha \,.\, z^2h^{2n}}{(h^{2n}z^2-z_1^2)(h^{2n}z^2-z_2^2)}$$

$$= \frac{\omega\tan^2\alpha}{\pi i(z_1^2-z_2^2)}\left[\frac{\sigma'}{\sigma}(u-u_1) - \frac{\sigma'}{\sigma}(u-u_2)\right] + \frac{\eta(u_1-u_2)}{\pi i(z_1^2-z_2^2)}\tan^2\alpha. \tag{10}$$

Hence $S_r(z)$ is determined, being expressed in terms of known σ and σ' functions and of given functions of known constants. For the period ω depends on the choice of the first arbitrary circle; $2a$ is the angle at which the two given circles intersect; z_1 and z_2 are the roots of a given quadratic equation (see quadratic equation under (4)); u, u_1, u_2, and ω' are respectively determined by the equations

$$z = e^{\frac{u\pi i}{2\omega}}, \qquad z_1 = e^{\frac{u_1\pi i}{2\omega}}, \qquad z_2 = e^{\frac{u_2\pi i}{2\omega}},$$

$$\sqrt{\frac{1-\sin a}{1+\sin a}} = h = e^{\frac{\omega'}{\omega}\pi i};$$

and the function η is determined by the equation

$$\eta = \frac{\pi^2}{2\omega}\left[\frac{1}{6} - \sum^n \frac{4h^{2n}}{(1-h^{2n})^2}\right].$$

See Weierstrass and Schwarz's Elliptic Function Formulas, § 6, (10).

The h introduced under (3) is compatible with h in (7); since in the Σ's (7) and those that follow, that they may be convergent, h must be < 1. (See Weierstrass and Schwarz, § 6, (3)). This restriction we have expressed under (2).

III.

Here it is only intended to suggest a plan, similar to that employed in I., by which the function $S_A(z)$ may be calculated. This may be done as follows: By making use of formulas (6), § 8, Weierstrass and Schwarz's Elliptic Function Formulas, also of (4), § 9, and taking note of the fact, as above, that when u becomes $u - u_1$, or $u - u_2$, z becomes z/z_1 or z/z_2 respectively, one can construct the function $S_A(z)$, which will have the form

$$C_1\,\mathrm{p}\,(u-u_1) + C_2\,\mathrm{p}\,(u-u_2) + C_3\frac{\sigma'}{\sigma}(u-u_1)$$

$$+ C_4\frac{\sigma'}{\sigma}(u-u_2) + C_5 = S_A(z). \qquad (11)$$

The C's being constants which are functions of known constants, z_1, z_2, u_1, u_2, etc.

GÖTTINGEN, *July* 3, 1891.

DAVIS'S LOGIC OF ALGEBRA.*

The present century has been prolific in its contributions to formal algebra—the philosophy that underlies the symbols and the operations of this science. The fraction, the negative, the incommensurable, and the imaginary have forced mathematicians to examine the foundations upon which they were building. The work of Grassman and of Hamilton broadened the field and by extending the meanings of the fundamental operations, quickened the interest in this examination.

Our American text-books have been slow to recognize the importance of these investigations. For this our technical schools and colleges have been somewhat to blame. They demand, before everything, that applicants and students shall be expert problem solvers, and little time is given to the consideration of the reasoning that underlies the algebraic processes. To be expert in the solution of problems is very desirable, but the *educational* value of algebra undoubtedly lies, not so much in this as in the practice which it might give in abstract thinking. To the great majority of men algebra as an *instrument* or *machine* is useless, but the practice it gives or ought to give in connected and accurate thought is of the greatest value to everyone.

In the book before us Mr. Davis has attempted, as we think successfully, to exhibit in a logical manner the fundamental laws that govern algebraic processes. He insists, at the outset, on the purely *formal* nature of the science. He elucidates clearly and succinctly the gradual extension of the meanings of the ordinary arithmetical operations of addition, division, etc. The pupil is gradually led from the consideration of operations performed on integers to the consideration of similar operations performed on fractions, on incommensurables, and on imaginaries.

The most commendable features, among many that are excellent, are his insistence on the "cut and try" nature of the inverse operations, and his full and careful treatment of incommensurables. On the other hand, we are disposed to think that the subject of imaginaries is easier of comprehension when presented in a somewhat different form.

The work is admirably adapted to students who already know their algebra as it is generally taught, and it is invaluable to teachers who wish to impress on their pupils the importance of the foundations of this branch of mathematics.

W. P. D.

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* An Introduction to the Logic of Algebra. By Ellery W. Davis, Ph. D. New York: John Wiley & Sons.

SOLUTIONS OF EXERCISES.

162

A BODY is projected at an angle of 30° with the horizon, with a given velocity. Determine the constant resistance it must suffer in the direction contrary to its motion in order that it may come to rest when it returns to the horizontal plane whence it started. Also determine the horizontal range, time of flight, and length of trajectory.

[*Jas. M. Ingalls.*]

SOLUTION.

Let V be the initial velocity, α the angle of projection, φ the inclination at a point of the trajectory where the velocity is v, U and u the corresponding *horizontal* velocities, r the constant resistance, and w the weight of the projectile. Put $r/w = n$. Then we have

$$\frac{du}{u} = n \frac{d\varphi}{\cos \varphi};$$

whence

$$\log u = n \log \tan \left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right] + C.$$

Let v_0 be the velocity when $\varphi = 0$, that is, at the summit of the trajectory; then $C = \log v_0$, and we have

$$u = v_0 \tan^n \left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right].$$

$$\therefore v_0 = u \cot^n \left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right] = U \cot^n \left[\frac{1}{4}\pi + \frac{1}{2}\alpha\right]. \qquad (1)$$

Substituting the above value of u in the differential equations

$$dt = -\frac{u}{g}\sec^2\varphi d\varphi,$$

$$dx = -\frac{u^2}{g}\sec^2\varphi d\varphi,$$

$$dy = -\frac{u^2}{g}\tan\varphi \sec^2\varphi d\varphi,$$

$$ds = -\frac{u^2}{g}\sec^3\varphi d\varphi,$$

integrating between the limits α and φ, substituting for v_0 its value from (1), and reducing, we have (supposing $n > 1$) the following general equations:

$$t = \frac{V}{g}\frac{n - \sin\alpha}{n^2 - 1} - \frac{v}{g}\frac{n - \sin\varphi}{n^2 - 1},$$

$$x = \frac{V^2}{g}\frac{(2n - \sin\alpha)\cos\alpha}{4n^2 - 1} - \frac{v^2}{g}\frac{(2n - \sin\varphi)\cos\varphi}{4n^2 - 1},$$

$$y = \frac{V^2}{g}\frac{(2n - \sin\alpha)\sin\alpha - 1}{4(n^2 - 1)} - \frac{v^2}{g}\frac{(2n - \sin\varphi)\sin\varphi - 1}{4(n^2 - 1)},$$

$$s = \frac{V^2}{g}\frac{2n(n - \sin\alpha) - \cos^2\alpha}{4n(n^2 - 1)} - \frac{v^2}{g}\frac{2n(n - \sin\varphi) - \cos^2\varphi}{4n(n^2 - 1)}.$$

From (1) we have

$$v = v_0 \sec\varphi \tan^n\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right],$$

which when $n > 1$, reduces to zero when $\varphi = -\frac{1}{2}\pi$. Therefore when the constant resistance is greater than the weight of the projectile, the velocity continually decreases, and becomes zero when the last element of the trajectory is vertical. At this point we have

$$t_{-\frac{1}{2}\pi} = \frac{V}{g}\frac{n - \sin\alpha}{n^2 - 1},$$

$$x_{-\frac{1}{2}\pi} = \frac{V^2}{g}\frac{(2n - \sin\alpha)\cos\alpha}{4n^2 - 1},$$

$$y_{-\frac{1}{2}\pi} = \frac{V^2}{g}\frac{(2n - \sin\alpha)\sin\alpha - 1}{4(n^2 - 1)},$$

$$s_{-\frac{1}{2}\pi} = \frac{V^2}{g}\frac{2n(n - \sin\alpha) - \cos^2\alpha}{4n(n^2 - 1)}.$$

To determine the relation that must exist between n and α, in order that the velocity of the projectile may be zero when it returns to the horizontal plane passing through the point of departure, we must make $y_{-\frac{1}{2}\pi} = 0$; which gives

$$\sin\alpha\,(2n - \sin\alpha) = 1,$$

or

$$n = \frac{1 + \sin^2\alpha}{2\sin\alpha}.$$

In exercise 162, $\alpha = 30°$; whence $n = \frac{5}{4}$. That is, the resistance is one and one-fourth the weight of the projectile. Substituting these values of α and n in the above equations, we have

$$T = \frac{4}{3}\frac{V}{g}; \qquad X = \frac{4\sqrt{3}}{21}\frac{V^2}{g}; \qquad S = \frac{2}{5}\frac{V^2}{g}.$$

For the summit we have from the general expression for y, making $\varphi = 0$,

$$y_0 = \frac{4\,v_0^2}{9g}.$$

But from (1) we have

$$v_0^2 = V^2 \cos^2 30° \cot^{\frac{8}{3}} 60° = \frac{V^2}{4 \times 3^{\frac{1}{3}}};$$

$$\therefore y_0 = \frac{1}{9 \times 3^{\frac{1}{3}}}\,\frac{V^2}{g}.$$

[*Jas. M. Ingalls.*]

282

THE centre and three points do not determine a conic when two of the three points are at the extremities of a diameter ($2k$). [*R. H. Graves.*]

SOLUTION.

If the centre be taken as the origin of co-ordinates, and $(+c, 0)$ and $(-c, 0)$ be two given points, the equation to the conic is

$$x^2 + Bxy + Cy^2 = c^2.$$

One more point cannot determine B and C. [*W. M. Thornton.*]

288

IF $\kappa, \lambda, \mu, \nu$ be any four angles, and if

$$\kappa + \lambda + \mu + \nu = 2\sigma,$$

prove

$$\begin{aligned}\sin\kappa \sin\lambda \sin\mu \sin\nu &+ \cos\kappa \cos\lambda \cos\mu \cos\nu \\ &= \sin(\sigma - \kappa)\sin(\sigma - \lambda)\sin(\sigma - \mu)\sin(\sigma - \nu) \\ &+ \cos(\sigma - \kappa)\cos(\sigma - \lambda)\cos(\sigma - \mu)\cos(\sigma - \nu).\end{aligned}$$

[*Yale.*]

SOLUTION.

Expressing $\sin\kappa \sin\lambda$, etc. as functions of sums and differences of angles, we obtain

$$\begin{aligned}\sin\kappa \sin\lambda \sin\mu \sin\nu &+ \cos\kappa \cos\lambda \cos\mu \cos\nu \\ &= \tfrac{1}{2}\cos(\kappa + \lambda)\cos(\mu + \nu) + \tfrac{1}{2}\cos(\kappa - \lambda)\cos(\mu - \nu).\end{aligned}$$

If we put $\varkappa + \lambda = 2\sigma - \mu - \nu$ and $\mu + \nu = 2\sigma - \varkappa - \lambda$, it is readily seen that this expression will not change its value by substituting $\sigma - \varkappa$ for $\varkappa$, etc.

[*Ormond Stone.*]

313

GIVEN three points and three straight lines in a plane, the determinant of the nine perpendiculars from the points to the lines is equal to twice the product of the areas of the triangles formed by the points and by the lines, divided by the radius of the circle circumscribing the latter.

[*W. W. Johnson.*]

SOLUTION.

Let the triangle formed by the three lines be the triangle of reference and the co-ordinates of the three points be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$. To prove

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \frac{2\Delta\Delta'}{R}.$$

Δ', the area of the triangle formed by the three points, is

$$\frac{1}{2S}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

where $S = \alpha \sin A + \beta \sin B + \gamma \sin C$ (see Whitworth's Modern Analytic Geometry, p. 22), $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, and $a\alpha + b\beta + c\gamma = 2\Delta$; whence the theorem.

[*H. B. Newson.*]

318

D is any point in the base BC of a triangle ABC. O_1 and O_2 are the centres of the inscribed circles of BAD and CAD. Then, if $\triangle =$ area ABC,

(1) Area $AO_1O_2 = \dfrac{r\triangle}{a}$ cosec ADC;

(2) If AD bisects A, $AO_1O_2 = \dfrac{r\triangle}{b+c}$ cosec $\frac{1}{2}A$;

(3) If AD is the altitude, $AO_1O_2 = \dfrac{r\triangle}{a}$.

[*T. U. Taylor.*]

SOLUTION.

(1) $AO_1 = \dfrac{c \sin \frac{1}{2}B}{\sin(\frac{1}{2}B + BAO_1)}$, $AO_2 = \dfrac{b \sin \frac{1}{2}C}{\sin(\frac{1}{2}C + CAO_2)}$;

$$\therefore \text{ area } AO_1O_2 = \tfrac{1}{2}AO_1 \,.\, AO_2 \sin O_1AO_2$$

$$= \tfrac{1}{2}\frac{bc \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin(\frac{1}{2}B + BAO_1)\sin(\frac{1}{2}C + CAO_2)}$$

$$= \frac{r\triangle}{2a \sin \frac{1}{2}ADC \cos \frac{1}{2}ADC}$$

$$= \frac{r\triangle}{a} \operatorname{cosec} ADC.$$

(2) If AD bisects A, then $ADC = 90° - \frac{1}{2}(C - B)$.
By substituting this in (1) it easily reduces to the required form.

(3) If AD is the altitude, $ADC = 90°$;

$$\therefore AO_1O_2 = \frac{r\triangle}{a}.$$

[*T. U. Taylor.*]

319

To construct a triangle ABC ($BC > AB > CA$), the angle A being known, and having given

(1) $AB + BC = m$, $AB + CA = n$;
or (2) $BC - AB = p$, $AB - CA = q$.

[*J. E. Hendricks.*]

SOLUTION.

1. Draw an indefinite right line MN, and through any point D draw PP', making an angle with MN equal to the given angle at A. From D toward M set off $DB = m$, and from B toward N set off $BF = n$. With D as centre and radius DF describe a semicircle EFE' and join EB and EF. With F as centre and radius DF describe an arc cutting EB in G, and join FG. Through B parallel with FG draw BC meeting EF produced in C, and through C draw CA parallel with PP' and intersecting BD in A; then is ABC the required triangle.

Because the triangle FDE is isosceles the similar triangle FAC is isosceles, therefore $AC = AF$. Through C draw CH parallel with AD and intersecting PP' in H. Then, by similar triangles, we have,

$$EF : FD : : EC : CH.$$

Also $$EF : FG : : EC : CB.$$

But $FG = FD$ by construction; $\therefore CB = CH = AD$.

2. Let MN and PP' be drawn as in 1. From D towards M take $DB' = p$, and from B' towards M take $B'F = q$. (Because $m - n = p + q$, F occupies the same position on the line MN in both cases.) With D as centre and radius DF describe the semicircle EFE', as in the construction of 1, and join $E'B'$ and EF. With F as centre and radius DF describe an arc cutting $B'E'$ in G', and join FG'. Through B' parallel with FG' draw $B'C'$ meeting EF produced in C' and draw $C'A$ parallel with PP' and intersecting DB in A; then is $AB'C'$ the required triangle.

Because the triangle FDE' is isosceles the similar triangle FAC' is isosceles, therefore $AC' = AF$. Through C' draw $C'H'$ parallel with AD and intersecting PP' in H'. Then, by similar triangles, we have,

$$E'F : FD :: E'C' : C'H'.$$

Also $$E'F : FG' :: E'C' : C'B'.$$

But $FG' = FD$ by construction; $\therefore$ $C'B' = C'H' = AD$.

[*J. E. Hendricks.*]

EXERCISES.

329

What relations must subsist between the lengths of the edges of a tetrahedron in order that the perpendiculars from the vertices to the opposite sides may meet in a common point? [*Yale Prize Problem.*]

330

Find the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots + [\tfrac{1}{2}n(n+1)]^2.$$

[*Artemas Martin.*]

331

The extremities of a diameter of a variable ellipse having fixed foci lie on a fixed hyperbola having the same foci; show that the extremities of the conjugate diameter lie on another hyperbola having the same foci.

[*W. Woolsey Johnson.*]

332

Four equianharmonic points give four triangles which have four circumcircles. Show that the inverses of any point with regard to these four circles are equianharmonic. [*Frank Morley.*]

333

Show that

$$\sin\theta > \theta - \frac{\theta^3}{3!} + \frac{1}{45}\left[\frac{\theta^5}{2^2} - \frac{\theta^7}{2^9} + \ldots (-)^{m+1}\frac{\theta^{2m+3}}{2^{\frac{1}{2}(m^2+9m+6)}} \pm \ldots\right];$$

the general term being the mth within the brackets. [*W. H. Echols.*]

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Find the necessary relation between the ten distances of five points in space. [*Yale Prize Problem.*]

CONTENTS.

ERRATUM.

Page 75, line 10, for $\pi - \theta$ read $\pi - 2\theta$.

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